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AN AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
SCIENTIFIC REPORT

LECTURE SERIES IN DIFFERENTIAL EQUATIONS

SESSION I
CONTROL THEORY

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GEORGETOWN UNIVERSITY
OCTOBER 2, 1965

49

AFOSR 65 - 1886

October 1965

An Air Force Office of Scientific Research

Scientific Report

LECTURE SERIES IN DIFFERENTIAL EQUATIONS

SESSION I

CONTROL THEORY

Georgetown University

October 2, 1965

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This Series is sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant Nr. AF AFOSR 984-66

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LINEAR CONTROL PROCESSES AND MATHEMATICAL PROGRAMMING

by

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This research was been partially supported by the National Science Foundation under Grant GP-2633 with the University of California.

Linear Control Processes and Mathematical Programming

by George B. Dantzig

Linear Control Process Defined [1], [2]:

We shall consider an "object" defined by its $n + 1$ coordinates $x = (x_0, x_1, \ldots, x_n)$ whose "motion" described as a function of a parameter "time" (t) can be written as a linear system of differential equations:

$$\frac{dx}{dt} = A^t x + B^t u \tag{1}$$

where A^t, B^t are known matrices that may depend on t and $u = (u_1, u_2, \ldots, u_p)$ is a control vector that must be chosen from a convex set, $u \in U(t)$ for every $0 \leq t \leq T$. The time period $0 \leq t \leq T$ is fixed and known in advance. The coordinate $x_0 = x_0(t)$ represents the "cost" of moving the object from its initial position to $x_0(t)$. For this purpose it may be assumed that $x_0(0) = 0$. Defining

$$\bar{x} = (0, x_1, x_2, \ldots, x_n) \tag{2}$$

the object is required to start somewhere in a convex domain $\bar{x}(0) \in S_0$ and to terminate at $t = T$ somewhere on another convex domain $\bar{x}(T) \in S_T$.

Problem: Find $u \in U(t)$ and boundary values $\bar{x}(0) \in S_0$, $\bar{x}(T) \in S_T$, such that $x_0(T)$ is minimized.

Assuming $u \in U(t)$ is known, the system of differential equations can be integrated to yield an expression for $X(T)$ in terms of $X(0)$ and $u \in U(t)$. This is true in general but will be illustrated for the case when A^t and B^t do not depend on t; in this case

$$(3) \qquad X(T) = e^{TA}X(0) + \int_0^T e^{(T-t)A}B\, u(t)\,dt$$

where $u(t) \in U(t)$ is a convex set and where we assume the integral exists whatever be the choice of the $u(t) \in U(t)$ for $0 \leq t \leq T$.

Generalized Linear Program [3]:

Our general objective is to illustrate how mathematical programming and, in particular, how the decomposition principle in the form of the generalized linear program can be applied to this class of problems. An elegant constructive theory emerges, [4].

A generalized linear program differs from a standard linear program in that the vector of coefficients, say P, associated with any variable μ need not be constant but can be selected from a convex set C. For example:

Problem: Find Max λ, $\mu \geq 0$ such that

$$(4) \qquad \begin{aligned} U_0\lambda + P\mu &= Q_0 \\ \mu &= 1 \end{aligned}$$

where U_0, Q_0 are specified vectors and $P \in C$ convex.

The method of solution assumes we have initially[1] on hand m particular choices $P_i \in C$ with the property that

$$(5) \qquad \begin{aligned} U_0\lambda + P_1\mu_1 + P_2\mu_2 + \ldots + P_m\mu_m &= Q_0 \\ \mu_1 + \mu_2 + \ldots + \mu_m &= 1 \end{aligned}$$

has a unique "feasible" solution; that is to say $\lambda = \lambda^0$, $\mu_i = \mu_i^0 \geq 0$ and the matrix

$$(6) \qquad B^0 = \begin{bmatrix} U_0 & P_1 & \ldots & P_m \\ 0 & 1 & \ldots & 1 \end{bmatrix}$$

is non-singular (i.e. the columns of B^0 form a basis). Because $P_i \in C$, the vector $P^0 = \Sigma\, P_i\mu_i^0$ constitutes a solution $P = P^0$ for (4) except that $\lambda = \lambda^0$ may not yield the maximal λ.

To test whether or not P^0 is an optimal solution one determines a row vector $\bar{\pi} = \bar{\pi}^0$ such that

$$(7) \qquad \bar{\pi}^0 B^0 = (1, 0, \ldots, 0)$$

and then a value δ and a vector $P_{m+1} \in C$ such that

$$(8) \qquad \delta = \bar{\pi}^0 P_{m+1} = \underset{P \in C}{\mathrm{Min}}\ \bar{\pi}^0 P$$

where we denote

---

[1] This is not a restrictive assumption since there is an analogous method for obtaining such a starting solution, see [4].

$$(9) \qquad \bar{P} = \begin{bmatrix} P \\ 1 \end{bmatrix}$$

If it turns out that $\delta = 0$, then $P = P^0$ is an optimal solution.

If P^0 is not optimal, system (5) is augmented by P_{m+1}. After one or several iterations k the augmented system takes the form of a linear program:

<u>Problem:</u> Find Max λ, $\mu_i \geq 0$

$$(10) \qquad U_0\lambda + \sum_{1}^{m+k} P_i\mu_i = Q_0$$

$$\sum_{1}^{m+k} \mu_i = 1$$

Letting B^k denote the basis associated with an optimal basic feasible solution $\mu_i = \mu_i^k$ to (10), π^k is defined analogous to (7) and δ^{k+1} and P_{m+k+1} analogous to (8). If it turns out that $\delta = 0$, the solution

$$(11) \qquad P^k = \sum_{1}^{m+k} P_i\mu_i^k$$

is optimal. If not the system is augmented by P_{m+k+1} and the iterative process is repeated.

It is known under certain general conditions such as C bounded and the initial solution is non-degenerate (i.e. $\mu_i^0 > 0$), that $\bar{\pi}^k \to \bar{\pi}^*$ and $P^k \to P^*$ on some subsequence k and that $P = P^*$ is optimal. The two fundamental properties of $\bar{\pi}^*$ are

$$(12) \qquad \bar{\pi}^* \neq 0 \text{ and } \bar{\pi}^*\bar{P} \geq \bar{\pi}^*\bar{P}^* = 0 \text{ for all } P \in C.$$

The entire process can be considered as constructive providing it is not difficult to compute the various P_{m+k+1} from (8) with $\bar{\pi} = \bar{\pi}^{m+k}$. Another point is that the iterative process terminates in a finite number of steps if C is a convex polyhedral set and P_{m+k} constitute extreme solutions from it. In all cases an estimate is available on how close the k^{th} solution is to an optimal value of λ.

<u>Application of the Generalized Program to the Linear Control Process:</u>

Let us denote

$$P = \int_0^T e^{(T-t)A}B\, u(t)dt \tag{13}$$

and note that P is an element of a convex set C_μ generated by choosing all possible $u(t) \in U(t)$. We specify that $U_0 = (1, 0, \ldots, 0)$, and denote by $\lambda = -X_0(T)$, where $X_0(T)$ is the coordinate of X(T) to be minimized. Then

$$X(T) = -U_0\lambda + \bar{X}(T) \tag{14}$$

We further define Q_0 by

$$\bar{X}(T) = e^{TA}X(0) + Q_0 \ . \tag{15}$$

Substitution of these into (3) formally converts[2] the integrated form of the control problem into a generalized linear program (4).

---

[2] Actually Q_0 is not given but is an element of a convex set. To simplify the discussion which follows we assume Q_0 is a fixed vector.

Each cycle of the iterative process yields a known row vector, which we partition

(16) $$\bar{\pi}^{k+1} = [\pi \ , \ \Theta]$$

where π represents it first $n + 1$ components corresponding to P and Θ its last component. Since π is known, our choice for P_{m+k+1} is

(17) $$\pi P_{m+k+1} = \text{Min} \left\{ \int_0^T \pi \ e^{(T-t)A_B} \ u(t)\,dt \right\}$$
$$= \int_0^T \left\{ \underset{u \in U(t)}{\text{Min}} \ \pi \ e^{(T-t)A_B} \ u(t) \right\} dt$$

where clearly the minimum is obtained when, in (16), the integrand for each t is selected to be minimum.

Note that

(18) $$\phi_{t,\pi} = \pi \ e^{(T-t)A_B}$$

is a row vector that can be computed for each t. For example, $\phi_{t,\pi}$ can be represented by a finite sum of vectors whose weights depend on t and the eigen values of A. The new extremal solution P_{m+k+1} is obtained by choosing the control which minimizes the linear form in u for each t ; i.e., find

(19) $$\text{Min}(\phi_{t,\pi} u) \ , \ u \in U(t) \ .$$

For example if U(t) is a polyhedral set then (19) is a linear program. If U(t) is the same for all t , then only the objective form, $\emptyset_{t,\pi}u$, varies for different t ; except for the objective form the linear programs are the same for all t .

If optimal π^* is used, then the optimal control u (except for a set of measure zero) satisfies

$$(20) \qquad \operatorname{Min}[\emptyset^*(t)u] \quad , \quad u \in U(t)$$

where $\emptyset^*(t) = \pi^* e^{(T-t)A}B$. Pontryagin refers to this as <u>the maximal principle</u>. It is, as we have just shown, also a consequence of the decomposition principle of linear programming.

<u>Conclusion:</u>

In our approach the general control obtained for each cycle is a linear combination of exactly n+1 special controls obtained by minimizing for each t, the linear expression (19) in u for n+1 choices of π . These special controls may be referred to as extreme controls. The latter each in themselves do not maintain feasibility, that is to say guarantee that the object will move from $\bar{X}(0)$ to $\bar{X}(T)$. Each new linear combination of these special controls will, however, generate a new feasible control with a lower value[3] for the total cost $X_0(T)$. Under conditions stated this iterative process is known to converge.

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3 If basic solution is non-degenerate.

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THE LURIE PROBLEM ON NONLINEAR CONTROLS

by

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The Lurie Problem On Nonlinear Controls

by Solomon Lefschetz

Some twenty five years ago the mathematical world was presented by the distinguished mathematical engineer Anatole Lurie [1] with a set of nonlinear control differential equations of great generality. In compact vector-matrix form [2] (with an improvement due to Malkin, Lefschetz and V. M. Popov) the system looked like this:

$$(1)\qquad \begin{cases} \dot{x} = Ax - b\varphi(\sigma) \\ \dot{\xi} = \varphi(\sigma) \\ \sigma = c'x - \gamma\xi \end{cases}$$

The initial uncontrolled system is

$$(1')\qquad \dot{x} = Ax$$

The notations are: x, b, c are n-vectors; A is a constant $n \times n$ matrix; greek letters represent scalars. $\varphi(\sigma)$ is the <u>characteristic</u> of the control and one assumes these properties: φ is continuous for all σ; $\varphi(0) = 0$, $\sigma\varphi(\sigma) > 0$ for $\sigma \neq 0$; $\Phi(\sigma) = \int_0^\sigma \varphi(\sigma)d\sigma$ (which is positive for $\sigma \neq 0$) $\to +\infty$ with $|\sigma|$.

The vector x is the <u>state</u> vector; b, c, γ are the control parameters; ξ is a control variable.

Roughly speaking assuming that the normal position for x is $x = 0$, given small deviations of x one hopes that by imbedding the system (1') into a suitable system (1) one makes $(x,\xi) \to 0$ hence $x \to 0$ as $t \to +\infty$, mathematically speaking in Lurie's monograph there is imbedded the following question now going by his name as

<u>Lurie's problem</u>. To find n.a.s.c. that the parameters b, c, γ, must satisfy in order that all solutions of (1) be asymptotically stable in the large whatever the choice of an admissible function $\varphi(\sigma)$. This is <u>absolute stability</u>.

This is the problem that I propose to discuss as a <u>mathematical problem</u>. Regarding its practical (R and D) value I must accept the evidence that a number of outstanding mathematical engineers have "lovingly" dealt with <u>the problem</u>:

Almost all the work done on the Lurie problem is an application of a fundamental theorem due to Piapunov plus a noteworthy complement due to Barbashin and Krassovskii - lumped in what I will refer to as the L.B.K. Theorem -- together with a highly significant observation made by J.P. LaSalle [3]. I must first describe these basic ideas.

<u>The L.B.K. Theorem</u>. <u>Let</u>

$$(2) \qquad \dot{u} = U(u), \quad U(0) = 0,$$

(u, <u>are n-vectors) be a real system, of class</u> C^1 <u>throughout the whole</u> n <u>space. A sufficient condition in order that every solution</u> u(t) <u>of</u> (2) $\to 0$ <u>as</u> $t \to +\infty$ <u>is the existence of a scalar function</u> V(u) <u>of class</u> C^1 <u>for all</u> u <u>with these properties</u>

(a) V(u) <u>is positive definite for all</u> u[> 0 for $u \neq 0$, $V(0) = 0$];

(b) $V(u) \to +\infty$ <u>with</u> $||u||$ (Barbashin-Krassovskii complement);

(c) <u>since V is of class</u> C^1 <u>for all</u> u <u>one may determine</u> $dV(u(t))/dt = \dot{V}(u)$ <u>along every solution</u> u(t) <u>of</u> (2)

and one must have $\dot{V}(u) < 0$ on every solution except $u = 0$.

(Class C^1: the function $f(u) = f(u_1,\ldots,u_n)$ is of class C^1 for all u if all the partials $\partial f/\partial u_n$ exist and are continuous for all u.)

We state LaSalle's complement as restricted to our "control" situation and merely remark that it has a much larger range of application.

LaSalle's complement. The L.B.K. Theorem still holds with $\dot{V} > 0$ replaced by $\dot{V} \leq 0$ under the condition that the set $\dot{V} = 0$ contains no other solution than $u = 0$.

Observation about Lurie's system. There are actually two distinct situations corresponding to $\gamma \neq 0$ and $\gamma = 0$.

I. $\gamma \neq$ The larger system is of dimension $n + 1$. This is indirect control. Its practical significance is that it operates through derivatives and hence makes possible use of a smaller, less heavy mechanism than the initial system: Minorsky's derivative control initiated this method.

II. $\gamma = 0$: direct control. The order of the system is unchanged. Actually in this case ξ plays no role and the true system is

$$(3) \qquad \dot{x} = Ax - b\varphi(c'x).$$

That is, controlling is obtained by adding a nonlinear part to the system.

Now if one assimilates (x,ξ) to the system variable Lurie's initial equation becomes

(4) $$\begin{cases} \dot{x} = Ax - b\varphi(c'x - \gamma\xi) \\ \dot{\xi} = \varphi(c'x - \gamma\xi) \end{cases}$$

which is like (3) but with A replaced by $\begin{pmatrix} A, & 0 \\ 0, & 0 \end{pmatrix}$ that is by a (necessarily) singular matrix. This has led many authors to consider (3) as the main system. Be as it may we will confine most of our discussion to (1) and indirect control ($\gamma \neq 0$).

Linearization. At all events if absolute stability is to be obtained the system will have to be at least asymptotically stable when $\varphi(\sigma) = \mu\sigma$, $\mu > 0$. It will then become linear and asymptotic stability will merely demand stability of the coefficient matrix. This attack was first persued and with full success by Yacubovich [5]. It is actually easier to deal first with the direct control (3). The linearized system is

(4') $$\dot{x} = (A - \mu bc')x$$

and so the matrix $A - \mu bc'$ must only have characteristic roots with negative real parts. The characteristic equation is $(zE - A + \mu bc')$ $= 0$ (E: unit matrix). If $E_z - A = A_z$ = characteristic matrix of A, the relation becomes $|A_z + \mu bc'| = 0$. A simple algebraic manipulation reduces this to the equivalent form

$$1 + \mu c' A_z^{-1} b = 0$$

Rather simple complex analysis yields the following results:

I. No characteristic root of A may lie to the right of the complex axis. Those on the complex axis must be at most double.

II. Same result for indirect control save that the root $z = 0$ must be at most simple.

III. Sufficient conditions for the asymptotic stability of the linearized indirect control for small μ are: A stable and $\underline{\gamma > 0}$. The preceding properties point out the relative simplicity of an indirect control with A stable (perhaps only feebly so). This is the situation that we propose to face in the general case. As a first step and since σ is the variable appearing in $\varphi(\sigma)$ it is convenient to replace by σ the independent variable ξ of (1). This is done with ease and yields the equivalent system

$$(5) \qquad \begin{cases} \dot{x} = Ax - b\varphi(\sigma) \\ \dot{\sigma} = c'Ax - \rho\varphi(\sigma) \\ \rho = c'b + \gamma \end{cases}$$

Absolute stability suggests recourse to the L.B.K. Theorem. A rather simple form of Liapunov function is obtained (from a generalization of one due to Lurie-Poshukov in the late forties) as

$$(6) \qquad V(x,\sigma) = x'Bx + \Phi(\sigma)$$

where the first term is a positive definite quadratic form that is B is a constant symmetric matrix symbolically described by $B > 0$. Thus V behaves in accordance with the L.B.K. Theorem. Then we find

$$(7) \qquad -\dot{V} = x'Cx + 2d'x\varphi + \rho\varphi^2$$

$$(8) \qquad -C = A'B + BA$$
$$d = Bb - \tfrac{1}{2} A'c$$

One wishes to have $-\dot{V} > 0$ for all x, σ and not both zero. This requires that $C > 0$. An early result of Liapunov asserts that given $C > 0$, there is a unique B solution of (8) and $B > 0$. Thus we merely need to remember $C > 0$.

Observing now that (7) is a quadratic form in the variables x, φ absolute stability will be achieved if one makes it positive definite in these variables. This may be expressed through two distinct modes of completing sequences.

(a) by leaving σ unchanged yielding the author's condition

$$(9) \qquad \rho > d'C^{-1}d \quad (\text{hence } \rho > 0);$$

(b) by modifying σ alone -- as done by Yacubovich -- leading to these conditions:

$$(10) \qquad \begin{cases} \rho > 0; \text{ existence of a B such that} \\ A'B + BA + \dfrac{d'd}{\rho} < 0. \end{cases}$$

Thus one may state.

__Theorem.__ __N.a.s.c.__ __to have__ $V, \dot{V}$ __satisfy the L.B.K. Theorem are equivalently__ $\{C > 0, (9)\}$ __or__ (10). __when this happens the system__ (1) (or 5) __is absolutely stable__.

We come now to the work of V.M. Popov [6] beyond doubt the most significant contribution to the Lurie problem since Lurie. The system dealt with is the indirect control (1) and the assumption continues to be that A is stable and $\gamma > 0$. Popov proves two central theorems. Stated with insignificant deviations from Popov they are:

Theorem I. A sufficient condition for the absolute stability of the indirect control (1) is the existence of two constants $\alpha > 0$, $\beta \geq 0$ such that for all real ω

$$(11) \qquad \mathrm{Re}\{(2\alpha\gamma + i\omega\beta)(c'A_{i\omega}^{-1}b + \frac{\gamma}{i\omega})\} \geq 0$$

We describe $\{\quad\} = P(\alpha,\beta,\omega)$ as the Popov function.

Theorem II. If absolute stability is determined by a Liapunov function of type "quadratic form in x, σ plus $\beta\Phi(\sigma)$" then the preceding theorem holds with this β and a suitable α (α maybe zero).

Actually Popov showed that the Liapunov function of Theorem II must have the form

$$V(x,\sigma) = x'Bx + \alpha(\sigma - c'x)^2 + \beta\Phi(\sigma)$$

where α, β are those which appear in the Popov function.

The state of Theorem I does not bring out the essential simplicity of the result. It is of special interest when $\alpha \neq 0$. Setting there $\frac{\beta}{2\alpha\gamma} = q$, one may replace $\underline{P}$ by $\underline{P}(q,\omega) = (1+i\omega q)(c'A_{i\omega}^{-1}b + \frac{\gamma}{i\omega}) = (1+i\omega q)(S_1(\omega) + iS_2(\omega))$.

Hence (11) becomes

$$S_1(\omega) - q\omega S_2(\omega) \geq 0.$$

This means that in the real x, y plane the curve

$$\Gamma : x = S_1(\omega), \quad y = \omega S_2(\omega)$$

has a tangent $y = qx$ through the origin in quadrants 1, 3 and is otherwise below that tangent. Since the functions $S_1(\omega)$, $S_2(\omega)$ are rational (hence Γ is a unicursal curve) the discovery of a tangent such as $y = qx$ is a rather simple matter -- much simpler than finding matrices C or B of our earlier conditions.

Open problem as yet unsolved: Is the Popov condition (11) necessary for absolute stability?

Remark about Popov's striking proof of Theorem I. It passed up Liapunov functions and replaced them by very advanced Fourier integral technique. Curiously a transfer function, à la linear theory, makes its appearance in the inequality (11). For Popov's function may be written

$$(2\alpha\gamma + \beta z)(c'A_z^{-1}b + \frac{\gamma}{z})]_z = i\omega$$

and

$$T(z) = c'A_z^{-1}\, b + \frac{\gamma}{z}$$

is the transfer function $\varphi(\sigma(t))$ to $\sigma(t)$ in (5).

From an R & D viewpoint a "sufficient condition" such as in Theorem I is definitely more important than a necessary one. The latter is perhaps more important as mathematics that as R & D information. It may be added also that the proof of Theorem II is far easier and simplier than that of Theorem I.

Hardly had Popov's expression (11) appeared on the scene, than a number of analogous results began to appear. They always involved some pair, V, $\dot{V}$ and in principle proceeded along much simpler lines than Popov's first theorem. We will just describe one very interesting result embodying a noted lemma of Yacubovich [7] strongly improved by Kalman [8] and still further advanced by Kenneth Meyer [9]. We will refer to it as the Y.K.M. lemma (Meyer's version with parts omitted.)

<u>The Y.K.M. Lemma</u>. <u>Assume</u> A <u>stable and let</u> b, k, <u>be real n-vectors and</u> τ <u>a non-negative constant. If</u>

$$\tau + 2 \operatorname{Re} k' A_{i\omega}^{-1} b \geq 0$$

<u>for all real</u> ω <u>then there exist two</u> $n \times n$ <u>matrices</u> $B > 0$, $D \geq 0$, <u>and a real n-vector</u> q <u>such that</u>

$$A'B + Ba = -qq' - D$$
$$Bb - k = \tau^{\frac{1}{2}} q.$$

The lemma with "> 0" rather than "$\geq$ 0" was proved by Yacubovich. Under the special assumption of complete controllability of (A,b) and complete observability of (A',c) it was extended (with new proofs) by Kalman. The complete controllability and observability conditions were recently removed by Kenneth Meyer.

If one takes

$$K = \frac{1}{2}\beta A'c + \alpha\gamma c, \quad \tau = \beta\rho = \beta\gamma + c'b$$

then the inequality of the lemma reduces to Popov's inequality (11).

One has then this result of Yacubovich and Kalman:

<u>Theorem. Popov's inequality with</u>

(a) $\alpha \geq 0, \quad \beta \geq 0, \quad \alpha + \beta > 0$

(b) $\tau > 0 \quad \text{or } \tau = 0, \quad \frac{1}{2}\beta A'c + \alpha\gamma c = 0, \ \alpha > 0$

<u>are n.a.s.c. for the existence of the Liapunov function of the second theorem with merely</u> $-\dot{V} \geq 0$.

Beyond the Lemma and under certain complicated complementary conditions given by Kalman one may show, using LaSalle's complement, that absolute stability is achieved.

If one takes Popov's inequality strictly as > 0 one may show that V and $-\dot{V}$ are both positive definite and hence, one has absolute stability.

Noteworthy work has been done recently by the mathematical engineer R. W. Brockett and his young associates (see[7]). Let $p(D)$, $q(D)$ be real polynomials of respective degrees n and at most n-1 in $D = \frac{d}{dt}$. With $\varphi(\sigma)$ as before the general problem which they have attacked is the scalar real differential equation

$$p(D)x + \varphi(\sigma) = 0, \quad \sigma = q(D)x. \tag{12}$$

and its absolute stability.

A description of one of his results -- his Theorem 4 -- will give an idea of his general procedure.

<u>Theorem. Let</u> $p(D) = Dh(D)$, <u>where all solutions of</u> $h(D)z = 0$ <u>are asymptotically stable. Then if there exists an</u> $r > 0$ <u>such that</u>

$m(z) = (1+rz)q(z)/p(z)$ has the property $\operatorname{Re} m(z) > 0$ when $\operatorname{Re} z > 0$, (12) is absolutely stable.

To indicate the nature of Brockett's Liapunov function for this case we need a few special notations.

(a) If $g(z)$ is a polynomial let $g_o(z)$ denote the polynomial made up of its even terms;

(b) if $g(z)$ is real, even, and such that $\beta(i\omega) \geq 0$ for all real ω, then (Wiener) $g(z)^{-} = k(z)k(-z)$, $k(z) \neq 0$ when $\operatorname{Re} z \neq 0$. Set $g(z) = k(-z)$;

(c) denote by X the real vector whose components are $x(t)$, $\dot{x}(t)$, $\ddot{x}(t),\ldots,x^{(n-1)}(t)$.

With integration in the space of X the form of Liapunov function given by Brockett is

$$V(X) = \int_{t(0)}^{t(X)} \{(\alpha+\beta D)q(D)z\cdot p(D)z + ((\alpha - \beta(D)q(-D)p(D))^{-}z)^2$$

$$+ \beta Dq(D)zf(q(D)z)\}dt$$

One may show that V is positive definite, $\dot{V}$ negative semi-definite for all admissible φ, then using LaSalle's complement one proves absolute stability. Technique of similar nature has been used, notably by Willems in an investigation of more restricted functions $\varphi(\sigma)$: of a differentiable monotone increasing $\varphi(\sigma)$. It may be pointed out that the first term of the expression for V(X) (term without φ) is essentially a quadratic form in X while the second term is really the same as the term $\Phi(\sigma)$ of the earlier Liapunov functions.

A number of authors have investigated an admissible class of characteristic functions $\varphi(\sigma)$ limited by an additional inequality $0 < \frac{\varphi(\sigma)}{\sigma} \leq \varkappa$ (finite). This has been dealt with at length in the recent book [8] by Aizerman and Gantmacher, which incidentally contains a very extensive bibliography. The Popov expression (11) is replaced by

$$P(\alpha, \beta, \omega, \varkappa) = P(\alpha, \beta, \omega) + \frac{2\alpha\gamma}{\varkappa} .$$

The modifications for indirect control are moderate, but not so for direct control, which require replacing the pair $(V, \dot{V})$ by (V, W) where W is a more restricted function then V, and depends largely on $\varkappa$.

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THE BANG-BANG PRINCIPLE

by

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This research was partially supported by NONR 3776-(00).

The Bang-Bang Principle

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1. <u>The physical significance of the bang-bang principle</u>. The bang-bang principle states that any response of a controlled system, which can be achieved by an arbitrary controller varying over the total control domain, can equally well be achieved by a controller which is restricted to the extreme values of the control domain. The term "bang-bang" refers to the abrupt switching of the controller from one of these extreme values to another. In engineering design it is often simpler to construct a control device with only a finite number of positions (say, the vertices of a polyhedron) rather than a continuum of possible positions (say, all the points of the solid polyhedron), and hence the bang-bang principle is of great practical importance - when it is applicable.

The bang-bang principle is not just a general principle but it is, in fact, a collection of precise mathematical theorems which center around a single physical concept. The basic mathematical result was obtained in relatively recent times (1940) by the Soviet scientist A. Liapounov [19], who thereby injected a new method - and also a new Liapounov - into control studies. The immediate application to control theory was presented by J. LaSalle [16] in a fundamental paper in

1960. Since that time a great deal of research has developed the applications of the bang-bang principle to control problems, both linear and nonlinear.

As a simple physical control process consider the mechanics of rowing a small boat across a smooth lake. The controlling force is produced by the two oars and the response is the heading and movement of the boat. If both oars are pulled simultaneously with equal force, the boat advances in a fixed direction. The direction can be changed in a controlled manner by using the oars together with each stroke but pulling one oar more strongly than the other. By controlling the difference in strength of the two oars, a continuum of control possibilities arises.

Now consider the same boat controlled by "bang-bang rowing". Here the oars act independently, but always with the fixed maximal pulling strength. The control in direction is effected by using one oar more frequently than the other. Thus the control force with each rowing stroke is always one of two extreme values (maximal right or maximal left), but any required heading of the boat can be achieved, even if not as smoothly as before.

Thus in the bang-bang control we replace a spatial variation of the controlling force (the resultant rowing stroke can range over a continuum of angular directions) by a temporal variation of the controlling force (the frequencies of left and right oar strokes). From this viewpoint the bang-bang principle resembles ergodic theory, although no precise interconnection is known relating these two disciplines.

2. <u>The bang-bang principle for linear processes.</u> Consider the first order vector differential system, or control process,

$$\mathcal{L}) \quad \dot{x} = Ax + Bu \; .$$

Here $x(t)$ is the real n-dimensional state vector at each instant of time t, and $u(t)$ is the real measurable control m-vector. The coefficients A and B are real constant matrices.

We fix the initial state x_0 in the real vector space R^n and choose various control functions $u(t)$ on $0 \leq t < \infty$ to determine the response $x(t)$ as the solution of the initial value problem

$$\frac{dx}{dt} = Ax + Bu(t) \; , \qquad x(0) = x_0 \; .$$

The controllers are arbitrary measurable functions with values restricted to a prescribed nonempty compact restraint set $\Omega \subset R^m$. For each time $t_1 \geq 0$ define the set of attainability $K(t_1)$ to consist of all endpoints $x(t_1)$ to responses initiating at x_0, for all possible controllers $u(t) \subset \Omega$ on $0 \leq t \leq t_1$.

<u>Definition.</u> Consider the linear control process in R^n

$$\mathcal{L}) \quad \dot{x} = Ax + Bu$$

with initial state x_0 at $t = 0$, and compact restraint set

$\Omega \subset R^m$. For each compact subset $Z \subset \Omega$ we define the set of attainability $K_Z(t_1)$ from x_0 by controllers $u(t) \subset Z$ on $0 \leq t \leq t_1$. We state that Z has the bang-bang property in case $K_Z(t_1) = K_\Omega(t_1)$ for all $t_1 \geq 0$.

If Ω is a compact convex set, then $K_\Omega(t_1)$ is also compact and convex. This follows from the variations of parameter formula for the response

$$x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-As}Bu(s)ds \ .$$

Since the map

$$u(\cdot) \to \int_0^{t_1} e^{-As}Bu(s)ds : L_\infty(0,t_1) \to R^n$$

is linear and compact, the properties of convexity and compactness of Ω are shared by $K_\Omega(t_1)$. The following theorem shows that $K_\Omega(t_1) = K_{H(\Omega)}(t_1)$, where $H(\Omega)$ is the convex hull of Ω . Hence $K_\Omega(t_1)$ is compact and convex even if Ω is an arbitrary compact set.

<u>Theorem 1</u>. <u>Consider the linear control process in</u> R^n

$$\mathcal{L}) \quad \dot{x} = Ax + Bu$$

<u>with initial state</u> x_0 <u>at</u> $t = 0$, <u>and compact restraint set</u> $\Omega \subset R^m$. <u>Let</u> Z <u>be a compact subset of</u> Ω <u>with the same convex hull</u>

$$H(Z) = H(\Omega) \ .$$

<u>Then</u> Z <u>has the bang-bang property</u> $K_Z(t_1) = K_\Omega(t_1)$ <u>for all</u> $t_1 \geq 0$.

The proof of theorem 1 was presented by L. Neustadt [22] and involves intricate functional analysis and the basic theorem of A. Liapounov, which will be discussed in Section 3.

<u>Corollary 1.</u> <u>Let</u> Z <u>be the boundary</u> $\partial\Omega$ <u>in</u> R^m. <u>Then</u> Z <u>has the bang-bang property</u>.

<u>Corollary 2.</u> <u>Let</u> Ω <u>be a convex polyhedron in</u> R^m, <u>and let</u> Z <u>be the set of vertices of</u> Ω. <u>Then</u> Z <u>has the bang-bang property</u>.

The theorem of LaSalle corresponds to Corollary 2 in the case where Ω is an m-cube.

A somewhat more general and very recent result [23] asserts that the set of extreme points of $H(\Omega)$ has the bang-bang property (even if this set is not compact).

The converse of Theorem 1 cannot hold in general. For suppose $B \equiv 0$, then any subset of Ω has the same (unique) response and so possesses the bang-bang property. In order to consider processes in which the controls have some reasonable effect we introduce the concept of a (completely) controllable process.

<u>Definition.</u> Consider the linear process in R^n

$$\mathcal{L})\quad \dot{x} = Ax + Bu$$

with no restraint on the controllers. Suppose for each pair of points x_0 and x_1 in R^n there exists a bounded measurable controller steering x_0 to x_1 on some $0 \leq t \leq t_1$. Then $\mathcal{L}$ is (completely) controllable.

It is well known [14] that $\mathcal{L}$ is controllable if and only if

$$\operatorname{rank}\,[B, AB, A^2B, \ldots, A^{n-1}B] = n\ .$$

The condition that the $n \times nm$ controllability matrix should have maximal rank n is "generic"; that is, every linear process can be approximated by controllable processes and also the property of controllability is maintained under perturbation. From this algebraic condition it follows that $\mathcal{L}$ is controllable if and only if $\dot{x} = -Ax - Bu$ is controllable, and this obtains just in case each initial state $x_0 \in R^n$ can be steered to the origin in a finite time duration by some bounded measurable controller. We prove a slight extension of this result.

<u>Theorem 2.</u> <u>Consider the linear process in</u> R^n

$$\mathcal{L})\ \dot{x} = Ax + Bu$$

<u>with initial state</u> x_0 at $t = 0$ <u>and compact restraint set</u> Ω <u>with interior in</u> R^m . <u>Then the set of attainability</u> $K(t_1)$ <u>is a convex body (with interior) for each</u> $t_1 > 0$ <u>if and only if</u>

$$\operatorname{rank}\,[B, AB, A^2B, \ldots, A^{n-1}B] = n\ .$$

<u>Proof.</u>

The set $K(t_1)$ is a compact convex set in R^n. First assume that the rank of the controllability matrix is less than n. Then there is a unit row n-vector v such that

$$vB = vAB = vA^2B = \ldots = vA^{n-1}B = 0$$

Using the Cayley-Hamilton theorem we compute

$$vA^kB = 0 \quad \text{for} \quad k = n, n+1, \ldots \; .$$

Hence

$$ve^{At}B = 0 \quad \text{for all } t \geq 0.$$

But $K(t_1)$ is the set of all endpoints

$$x(t_1) = e^{At_1}x_0 + e^{At_1}\int_0^{t_1} e^{-As}Bu(s)ds$$

for controllers $u(t) \subset \Omega$ on $0 \leq t \leq t_1$. Thus

$$v[K(t_1) - e^{At_1}x_0] = 0$$

and so $K(t_1)$ is a translate of a subset which lies in the hyperplane orthogonal to v. Therefore $K(t_1)$ has no interior points.

Conversely assume that the controllability matrix has rank n. Suppose that $K(t_1)$ has no interior and that there is a unit vector v for which

$$v[K(t_1) - e^{At_1}x_0] = 0 \text{ , for some } t_1 > 0 \text{ .}$$

In this case

$$\int_0^{t_1} v\, e^{A(t_1-s)} B(u_0 + u(s))ds = 0$$

where u_0 is a constant in the interior of Ω and $u(s)$ is an arbitrary controller near zero. Then, for $u(s) = 0$ we obtain

$$\int_0^{t_1} v\, e^{A(t_1-s)} B\, u_0\, ds = 0$$

and so

$$\int_0^{t_1} v\, e^{A(t_1-s)} B\, u(s) ds = 0 .$$

Since this equality holds for all small controllers,

$$v\, e^{A(t_1-s)} B \equiv 0 \quad \text{on } 0 \leq s \leq t_1 .$$

Set $s = t_1$ to get $vB = 0$. Next differentiate with respect to s and set $s = t_1$ to get $v\,AB = 0$. Continue in this way to obtain

$$vB = vAB = vA^2B = \ldots = vA^{n-1}B = 0$$

But this contradicts the hypothesis that the n rows of the controllability matrix are independent. Therefore $K(t_1)$ is a convex body. Q.E.D.

<u>Corollary</u>. The linear process $\mathcal{L}$ is controllable in R^n (with no control restraint) if and only if

$$\text{rank } [B, AB, A^2B, \ldots, A^{n-1}B] = n .$$

<u>Proof</u>.

Assume that the controllability matrix has rank n and take the unit m-cube centered at the origin as the restraint set Ω. Then, using controllers in Ω on $0 \leq t \leq 1$, we can steer the origin $x_0 = 0$ to any point x_1 in a neighborhood

N. By the linearity of $\mathcal{L}$, controllers in $k\Omega$ steer x_0 to all points in kN, for $k = 1,2,3,\ldots$. Thus x_0 can be steered to any point $x_1 \in R^n$ by a bounded controller $u(t)$ on $0 \leq t \leq 1$. Since this same conclusion holds for

$$-\mathcal{L}) \quad \dot{x} = -Ax - Bu ,$$

we can reverse the time sense and steer x_1 to $x_0 = 0$. Hence $\mathcal{L}$ is controllable.

Next assume that the rank of the controllability matrix is less than n. In this case there is a nonvanishing vector v such that

$$vB = vAB = vA^2B = \ldots = vA^{n-1}B = 0 .$$

But this implies that $ve^{At}B = 0$ and so

$$v \int_0^{t_1} e^{A(t_1-s)} Bu(s)ds = 0 .$$

Hence the origin $x_0 = 0$ can be steered only to points in the hyperplane orthogonal to v, and $\mathcal{L}$ fails to be controllable. Q. E. D.

<u>Theorem 3.</u> <u>Consider the controllable process in</u> R^n

$$\mathcal{L}) \quad \dot{x} = Ax + Bu$$

<u>with initial state</u> x_0 <u>at</u> $t = 0$, <u>and compact restraint set</u> Ω <u>with interior in</u> R^m. <u>Assume rank</u> $B = m$. <u>Then a compact subset</u> $Z \subset \Omega$ <u>has the bang-bang property if and only if</u> $H(Z) = H(\Omega)$.

Proof.

By theorem 1 a compact subset Z with $H(Z) = H(\Omega)$ has the bang-bang property, and we prove the converse under the above hypotheses.

Assume $H(Z) \neq H(\Omega)$ so that there exists a point in $H(\Omega)$ which is separated by a hyperplane from $H(Z)$. Thus there exists a supporting hyperplane π to $H(\Omega)$ such that π fails to meet $H(Z)$.

Now let P_0 be a boundary point of $K_\Omega(t_1) = K_{H(\Omega)}(t_1)$, for any fixed $t_1 > 0$. Let η_0 be any external unit (row) vector normal to a supporting hyperplane to $K_\Omega(t_1)$ at P_0. Then any controller $u_0(t) \subset H(\Omega)$ on $0 \leq t \leq t_1$ which steers x_0 to P_0 necessarily satisfies

$$\eta_0[e^{At_1}x_0 + e^{At_1}\int_0^{t_1} e^{-As}B\, u_0(s)ds]$$

$$\geq \eta_0[e^{At_1}x_0 + e^{At_1}\int_0^{t_1} e^{-As}Bu(s)ds] ,$$

where $u(t) \subset H(\Omega)$ is an arbitrary controller. Thus $u_0(t)$ satisfies the maximal principle [4]

$$\eta_0\, e^{At_1} e^{-At}\, Bu_0(t) = \max_{u \in H(\Omega)} \eta_0\, e^{At_1}\, e^{-At}\, Bu ,$$

almost always on $0 \leq t \leq t_1$.

Consider the linear map of R^n into R^m

$$\eta \rightarrow \eta B .$$

Since rank $B = m$, we can find a (row) vector η_1 such that

$\eta_1 B$ is along the outward normal to the hyperplane π. This means that

$$\max_{u \in H(\Omega)} \eta_1 e^{At_1} e^{-At} Bu$$

assumes its maximum, for each t near t_1, only on $H(\Omega) - H(Z)$.

Let $P_1 \in \partial K_\Omega(t_1)$ be a point where η_1 is an outward directed normal to $K_\Omega(t_1)$. Then P_1 cannot be attained by any controller restricted to $H(Z)$, since any such controller cannot satisfy the necessary maximal principle. Therefore $K_Z(t_1) \neq K_\Omega(t_1)$ and Z fails to have the bang-bang property. Q.E.D.

Corollary. *If Ω is also convex, then a compact $Z \subset \Omega$ has the bang-bang property if and only if Z contains all the extreme points of Ω.*

3. A. Liapounov's theorem and some generalizations. In this section we present a discussion of Liapounov's theorem on the convexity of the range of a vector measure.

Let $y(t)$ be a bounded measurable n-vector function on $0 \leq t \leq 1$. Let $\mathcal{B}$ be the σ-algebra (Borel σ-field) of all Lebesgue measurable subsets of $0 \leq t \leq 1$.

Theorem 4 (A. Liapounov). *For each set $E \in \mathcal{B}$ consider the point in R^n*

$$x_E = \int_E y(t)dt ,$$

and let K be the set of all such points x_E. Then K is convex and compact.

Proof (sketch).

First construct a continuous family of sets $D_\alpha \in \mathcal{B}$ $0 \leq \alpha \leq 1$ with $D_{\alpha_1} \subset D_{\alpha_2}$ if and only if $\alpha_1 \leq \alpha_2$, and the Lebesgue measure of D_α is

$$\mu(D_\alpha) = \alpha \ .$$

Such a continuous family is easily obtained [8] as a maximal, linearly ordered (by inclusion) chain of sets in $\mathcal{B}$, with the aid of the axiom of choice.

We use D_α and the first component $y_1(t)$ of $y(t)$ to construct a σ-algebra $\mathcal{A}_1 \subset \mathcal{B}$ whereon

$$\int_E y_1(t)dt = \mu(E) \int_0^1 y_1(t)dt \ .$$

To construct $\mathcal{A}_1$ first take $E_1 \in \mathcal{B}$ whereon

$$\int_{E_1} y_1(t)dt = \frac{1}{2} \int_0^1 y_1(t)dt, \quad \mu(E_1) = \frac{1}{2} \ .$$

The existence of E_1 follows from the properties of the family D_α. Namely,

$$\mu(D_\alpha - D_{\alpha-1/2}) = 1/2 \text{ on } 1/2 \leq \alpha \leq 1$$

and the integral of $y_1(t)$ over $(D_\alpha - D_{\alpha-1/2})$ is a continuous function $\phi(\alpha)$ such that

$$\frac{\phi(1) + \phi(1/2)}{2} = 1/2 \int_0^1 y_1(t)dt \ .$$

Thus for some intermediate α_1 on $1/2 \leq \alpha \leq 1$ we obtain $\phi(\alpha_1) = 1/2 \int_0^1 y_1(t)dt$, as required.

Next partition E_1 and $E_2 = [0,1] - E_1$ into two appropriate subsets E_3, E_4 and E_5, E_6 on each of which

$$\int_E y_1(t)dt = \frac{1}{4}\int_0^1 y_1(t)dt \ , \ \mu(E) = 1/4 \ .$$

Continue this partitioning to obtain a countable collection of such sets E and then let $\mathcal{A}_1$ be the σ-algebra generated by all these sets. Since $\int_E y_1(t)dt$ and $\mu(E)$ are each signed measures defined on $\mathcal{A}_1$, and since they are equal (up to a constant factor) on the algebra generated by the above countable family E_1, E_2, E_3, E_4, E_5, E_6, ... , we obtain

$$\int_E y_1(t)dt = \mu(E) \int_0^1 y_1(t)dt \ ,$$

for all $E \in \mathcal{A}_1$.

Next repeat the above argument to select σ-algebras $\mathcal{A}_n \subset \mathcal{A}_{n-1} \subset \ldots \subset \mathcal{A}_1 \subset \mathcal{B}$ such that

$$\int_E y(t)dt = \mu(E) \int_0^1 y(t)dt$$

for $E \in \mathcal{A}_n$.

Finally we prove the convexity of the set K. Suppose F_1 and F_2 are in $\mathcal{B}$ with

$$\int_{F_1} y(t)dt = a_1 \ , \ \int_{F_2} y(t)dt = a_2 \ ,$$

and consider the intermediate point

$$\lambda a_1 + (1-\lambda)a_2 \ , \ 0 < \lambda < 1 \ .$$

Consider the 2n-vector

$$y^*(t) = \begin{bmatrix} y(t)\ \chi_1(t) \\ y(t)\ \chi_2(t) \end{bmatrix}$$

with the characteristic functions $\chi_1(t)$ of F_1 and $\chi_2(t)$ of F_2. Let $\mathcal{A} \subset \mathcal{B}$ be a σ-algebra whereon

$$\int_E y^*(t)dt = \mu(E) \int_0^1 y^*(t)dt = \mu(E) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} .$$

Let D_α with $\mu(D_\alpha) = \alpha$ be a continuous family of sets in $\mathcal{A}$ and define

$$F = (D_\lambda \cap F_1) \cup [([0,1] - D_\lambda) \cap F_2] .$$

Then

$$\int_F y(t)dt = \int_{D_\lambda} y(t)\chi_1(t)dt + \int_{[0,1]-D_\lambda} y(t)\chi_2(t)dt$$

so

$$\int_F y(t)dt = \lambda a_1 + (1-\lambda)a_2 ,$$

and hence K is convex.

The compactness follows from arguments of functional analysis which we shall not indicate here. Q.E.D.

Remarks. Let us indicate the proof of the convexity of $K_\Omega(t_1)$, as in theorem 1. Let $u_a(t)$ and $u_b(t) \subset \Omega$ on $\mathcal{J}$: $0 \leq t \leq t_1$ be two controllers with responses $x_a(t)$ and $x_b(t)$ initiating at x_0 when $t = 0$. For each measurable subset $D \subset \mathcal{J}$ consider the real 2n-vector

$$w(D) = \begin{cases} \int_D e^{-As} \, B \, u_a(s)ds \\ \int_D e^{-As} \, B \, u_b(s)ds \end{cases} .$$

The vector-valued set function $w(D)$ has values

$$w(\mathcal{J}) = \begin{bmatrix} r_a \\ r_b \end{bmatrix} \text{ and } w(\emptyset) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

By Liapounov's theorem there is a set $D_\lambda \subset \mathcal{J}$ for which

$$w(D_\lambda) = \lambda \begin{bmatrix} r_a \\ r_b \end{bmatrix} \text{ and } w(\mathcal{J} - D_\lambda) = (1-\lambda) \begin{bmatrix} r_a \\ r_b \end{bmatrix} .$$

Define the controller, for $0 \leq \lambda \leq 1$,

$$u_\lambda(t) = \begin{cases} u_a(t) & \text{for } t \in D_\lambda \\ u_b(t) & \text{for } t \in \mathcal{J} - D_\lambda \end{cases} .$$

Then the corresponding response leads to

$$x_\lambda(t_1) = e^{At_1} x_0 + e^{At_1} \left[\int_{D_\lambda} e^{-As} Bu_a(s)ds + \int_{\mathcal{J}-D_\lambda} e^{-As} Bu_b(s)ds \right]$$

$$x_\lambda(t_1) = \lambda x_a(t_1) + (1-\lambda)x_b(t_1) .$$

Thus $K_\Omega(t_1)$ is convex for an arbitrary compact restraint set Ω.

We note that the bang-bang principle, say in the case of a convex polyhedron Ω, merely asserts that the required control can be attained by a controller resting at each vertex of Ω for some measurable duration of time. It is of great interest to find bang-bang controllers which are piecewise constant; which have only a finite number of switches rather than a complicated measurable switching set. Such results have been obtained by H. Halkin [9, 10, 11] and N. Levinson [18]. In their analyses the coefficient matrices $A(t)$ and $B(t)$ can vary analytically with time and need not be constant. For arbitrary integrable coefficient matrices the bang-bang principle holds, but the switches may well be necessarily infinite.

Another interpretation of the bang-bang principle restricts the controllers to be extreme functions rather than functions with values in the set of extreme points of Ω. That is, let Ω be a compact convex set in R^m with extreme point set W. Then theorem 1 asserts that $\overline{W}$ has the bang-bang property,

$$K_{\overline{W}}(t_1) = K_{\Omega}(t_1) \text{ for } t_1 > 0 .$$

The set M_Ω of all measurable controllers (almost always) in Ω is a weak * compact, convex set in the space M of all essentially bounded measurable m-vector functions on $\mathcal{J}$: $0 \leq t \leq t_1$. A theorem of S. Karlin [15] asserts that the extreme points of the set M_Ω are among the controllers having values in $\overline{W}$. If $W = \overline{W}$, the extreme points of M_Ω are precisely the controllers in $\overline{W}$.

4. Nonlinear bang-bang phenomena. Consider the nonlinear control process

$$\mathcal{S}) \quad \dot{x} = f(x,u)$$

where $f(x,u) \in C'$ in R^{n+m}. The state $x(t)$ at each time t is a real n-vector and the control $u(t)$ is a measurable m-vector function with values in some restraint set $\Omega \subset R^m$. We seek to steer an initail state x_0 to the origin $x_1 = 0$ in some finite time interval $0 \leq t \leq t_1$ by a controller in Ω. Since the process is nonlinear, the response $x(t)$ to a controller $u(t) \subset \Omega$ on $0 \leq t \leq t_1$ might not exist for the entire duration, and hence it is reasonable to consider just strictly local control problems for short time durations.

__Example.__ Consider the scalar process in R'

$$\dot{x} = u + u^2$$

with restraint Ω: $|u| \leq 2$. Then, since the linear approximation near $x = u = 0$ is the controllable process $\dot{x} = u$, each point x_0 near $x_1 = 0$ can be steered to the origin by a controller in Ω. However, if we use only bang-bang controllers with $|u(t)| = 2$, then $\dot{x}(t) > 0$ and so points $x_0 > 0$ cannot be steered to the origin.

The difficulties illustrated by the above example are resolved by the following theorem [20,21].

__Theorem 5.__ __Consider the control process in__ R^n

$$\mathcal{S}) \quad \dot{x} = f(x,u) \text{ in } C' \text{ in } R^{n+m}.$$

__Assume__

1) $f(0,0) = 0$

2) rank $[B, AB, A^2B, \ldots, A^{n-1}B] = n$

where $A = \dfrac{\partial f}{\partial x}(0,0)$ and $B = \dfrac{\partial f}{\partial u}(0,0)$.

Let π be a fixed convex polytope with the origin in its interior in R^m. Then there exists an $\varepsilon > 0$ and a neighborhood N of $x_1 = 0$ such that:

each initial state $x_0 \in N$ can be steered to $x_1 = 0$ by a measurable controller $u(t)$ on $0 \leq t \leq 1$ with values only in the finite set of vertices of the similar polytope $\varepsilon\pi$.

Finally let us turn to the nonlinear global control problem with bang-bang controllers. To insure global stability about the origin $x_1 = 0$ we impose the classical hypothesis of A. M. Liapounov. The bang-bang analysis is treated by the methods of A. Liapounov. In this manner we are able to obtain the following result.

Theorem 6. (Liapounov-Liapounov). Consider the control process in R^n

$$\mathcal{S}) \quad \dot{x} = f(x,u) \text{ in } C' \text{ in } R^{n+m}.$$

Assume that there exists a real function $V(x)$ in C' in R^n such that:

1) $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$

2) $\lim\limits_{|x| \to \infty} V(x) = \infty$

3) $\dfrac{\partial V}{\partial x^i} f^i(x,0) < 0$ for $x \neq 0$.

Also assume

4) $f(0,0) = 0$

5) $\text{rank}\,[B, AB, A^2B, \ldots, A^{n-1}B] = n$

where $A = \frac{\partial f}{\partial x}(0,0)$ and $B = \frac{\partial f}{\partial u}(0,0)$.

Then there exists an $\varepsilon > 0$ such that every initial point $x_0 \in R^n$ can be steered to $x_1 = 0$ in a finite time by a measurable controller $u(t)$ each of whose components takes on only the three values $+\varepsilon$, $-\varepsilon$, and 0 .

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MAR 11 1966

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Georgetown University Department of Mathematics Washington, D. C. 20007	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

LECTURE SERIES ON DIFFERENTIAL EQUATIONS
SESSION 1: CONTROL THEORY

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Scientific Interim

5. AUTHOR(S) *(First name, middle initial, last name)*

George B. Dantzig, Solomon Lefschetz, Lawrence Markus

6. REPORT DATE November 1965	7a. TOTAL NO. OF PAGES 48	7b. NO. OF REFS 37

8a. CONTRACT OR GRANT NO. AF-AFOSR 984-66 new	9a. ORIGINATOR'S REPORT NUMBER(S) No. 1
b. PROJECT NO. 9749-02	
c. 6144501F	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*
d. 681304	AFOSR 65-1886

3. Each transmittal of this document outside the agencies of the U. S. Government must have prior approval of AFOSR (SRGD).

11. SUPPLEMENTARY NOTES TECH, OTHER	12. SPONSORING MILITARY ACTIVITY Air Force Office of Scientific Research (SR 1400 Wilson Boulevard Arlington, Virginia 22209

13. ABSTRACT

This Scientific report summarizes the three lectures presented at the First Session of the Lecture Series in Differential Equations, sponsored by the AFOSR, and the Graduate Consortium of American, Catholic, Georgetown, George Washington and Howard Universities of the District of Columbia and the University of Maryland, and held at Georgetown University, 2 October 1965.

The First Session, on Control Theory, included lectures by Professors G. B. Dantzig, S. Lefschetz and L. Markus. Professor Dantzig illustrated how mathematical programming, and in ~~a~~ particular a generalized linear program, can be applied to a linear control process. The "problem" takes the form of minimizing the "cost" of moving an "object" from one convex domain to another by proper choice of a control vector and boundary conditions.

Professor Lefschetz dealt with the Lurie problem on nonlinear controls, i.e., the determination of necessary and sufficient conditions such that all solutions of a set of general nonlinear control differential equations are absolutely stable in the large whatever the choice of the admissible (function) characteristic of the control.

Professor Markus discussed the "bang-bang" theory of control as a physical concept and as a collection of precise mathematical theorems. The bang-bang principal states that any response of a controlled system which can be achieved by an arbitrary controller restricted to the extreme values of the control domains. Theorems presented relate both to linear and nonlinear control processes.

DD FORM 1 NOV 65 1473

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
ordinary differential equations linear ordinary differential equations nonlinear ordinary differential equations control theory mathematical programming "bang-bang" theory of control						